Conditional central limit theorem for subcritical branching random walk

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Outline

1. Branching Random Walk (BRW)
2. Main result: Conditional central limit theorem
3. Sketch of the proof
4. Applications: Conditional limit theorems

Branching Random Walk (BRW)

Given a probability measure $\{p_k\}_{k\geq 0}$ on $\mathbb{N}$ (called offspring distribution) and a probability measure G on $\mathbb{R}$, the branching random walk is defined as follows.

- At time 0, there is one particle positioned at 0, denoted by Z_0.
- It splits into a random number of particles according to $\{p_k\}_{k\geq 0}$, meanwhile, each of the particles are positioned independently (with respect to their parent) according to the same probability measure G, denoted by Z_1.
- Similarly, each particle in Z_1 splits independently according to $\{p_k\}_{k\geq 0}$ and is positioned independently according to G, which forms Z_2.
- And so on.

Assume: the reproduction and displacement mechanisms are independent.

Branching Random Walk (BRW)

We use the classical Ulam–Harris notation for discrete trees.

- $\mathbb{T}$: the genealogical Galton–Watson tree of this system rooted at $\varnothing$.
- $V(u)$: the position of u.
- $|u|$: the generation of u.
- Branching random walk: the random measure $\boldsymbol{Z_n = \sum_{u\in\mathbb{T}:|u|=n} \delta_{V(u)}}$.
- **$Z_n(A) := \#\{u \in \mathbb{T} : V(u) \in A, |u| = n\}$: the number of particles in the n-th generation located in A.**
- $N_n := Z_n(\mathbb{R})$: the size of n-th generation. Then $\{N_n; n \geq 0\}$ is a Galton-Watson process (GW process) with offspring distribution $\{p_k\}_{k\geq 0}$, which is called supercritical, critical, and subcritical according to $m := \sum kp_k > 1, \ = 1$ and < 1, respectively.
- $f(s) := \sum_k p_k s^k, \ \ s \in [0, 1]$. It is easy to obtain the generating function of N_n given by the iterate

$$f_n(s) = f\left(f_{n-1}(s)\right), \quad f_0(s) = s, \quad f_1(s) = f(s).$$

Known results

- The intensity measure of Z_n is $m^n G^{n*}$, where G^{n*} is the n-fold convolution of G.
- If G has mean μ and variance $\sigma_0^2 < \infty$, then for any $x \in \mathbb{R}$,

$$m^{-n}\mathbf{E}\left[Z_n((-\infty, \sqrt{n}\sigma_0 x + n\mu])\right] = G^{n*}(\sqrt{n}\sigma_0 x + n\mu) \to \Phi(x), \quad n \to \infty,$$

where $\Phi(x)$ is the standard normal distribution function.

Known results

Supercritical BRW ($m > 1$):

- Harris (1963) first conjectured the following central limit theorem (CLT) for the supercritical BRW: if G has mean zero and variance one,

$$\boldsymbol{m^{-n} Z_n((-\infty, \sqrt{n}x]) \to W\Phi(x)} \quad \textbf{in probability}, \quad \boldsymbol{n \to \infty}, \tag{1}$$

  where W is the limit of the additive martingale $\{m^{-n}N_n\}$ in GW process.
- Stam (1966), Kaplan and Asmussen (1976): under the assumption"particles' displacement are independent of their reproduction", and obtained that the convergence holds **almost surely**.
- Klebaner (1982), Biggins (1990): **removed** the assumption, and extended these results to the branching random walk in **time-varying environment**.

Known results

In subcritical case, the processes will die out with probability one.

Subcritical branching Brownian (BBM) motion with absorption:

- Liu (2021): obtained a Yaglom type asymptotic result for subcritical BBM with absorption.

Subcritical BRW ($m < 1$):

- (1) holds almost surely.
- Q: What's the conditional central limit theorem for subcritical BRW?

Assumption & Question

Assumption

- the reproduction and displacement mechanisms are independent.
- $0 < m < 1,\ \sigma^2 := \mathrm{Var}\, N_1 < \infty.$
- $\int_{\mathbb{R}} x\,\mathrm{d}G(x) = 0,\ \int_{\mathbb{R}} x^2 \mathrm{d}G(x) = 1.$

Question

In subcritical BRW, what's the asymptotic result of $\mathcal{L}\left(Z_n((-\infty, \sqrt{n}x]) \mid N_n > 0\right)$ under the assumptions as $n \to \infty$?

Previous results for subcritical GW process

Yaglom's theorem

For $0 < m < 1$, then

$$\mathcal{L}(N_n \mid N_n > 0) \xrightarrow{w} \mathcal{L}(\xi),$$

where $\xrightarrow{w}$ means weak convergence, and we say random variable ξ is the **Yaglom limit** of GW process $\{N_n; n \geq 0\}$.

- Yaglom (1947): showed that such limit exists when $m < 1$ and N_1 has a finite second moment.
- Heathcote, et al. (1967), Joffe (1967), Athreya and Ney (1972): generalized to the case without the second moment assumption.

Previous results for subcritical GW process

More generally, it is meaningful to extend to condition on $\{N_{n+l} > 0\},\ l = 1, 2, \cdots$.

Conditional limit theorem (Athreya and Ney, 1972)

For $0 < m < 1$, then

$$\lim_{n\to\infty} \mathbf{P}\left(N_n = j \mid N_{n+l} > 0\right) = b_j(l) \geq 0, \quad j \geq 1,$$

and $\sum_{j\geq 1} b_j(l) = 1$, its generating function is $m^{-l}[\mathscr{B}(s) - \mathscr{B}(sf_l(0))]$, where $\mathscr{B}(s)$ is the generating function of Yaglom limit.

Previous results for subcritical GW process

As $l \to \infty$, the limit distribution of $\mathcal{L}\left(N_n \mid N_{n+l} > 0\right)$ determines a Markov process whose n-step transition probabilities are given by

$$Q_n(i,j) = \lim_{l\to\infty} \mathbf{P}\left(N_n = j \mid N_{n+l} > 0, N_0 = i\right),$$

and we call the above Markov process is the Q-**process** associated with $\{N_n; n \geq 0\}$.

The positive recurrence for Q-process

For $0 < m < 1$, the Q-process is positive recurrent if and only if $\sum_k (k\log k)p_k < \infty$. In the positive recurrent case the **stationary measure** for Q is

$$\pi_j = \varphi(0) j b_j, \quad j = 1, 2, \cdots,$$

where $b_j = \mathbf{P}(\xi = j)$, $\varphi(0) = \lim_{n\to\infty} m^{-n}\mathbf{P}(N_n > 0)$. Furthermore, the generating function of $\{\pi_j\}_{j\geq 1}$ is $\varphi(0)s\mathscr{B}'(s)$.

- Joffe (1967), Athreya and Ney (1972): studied the positive recurrence for Q-process.
- **Remark:** Heathcote, et al. (1967), Athreya and Ney (1972) showed that $\varphi(0) > 0$ if and only if $\sum_k (k\log k)p_k < \infty$.

Main result

Theorem 1 (Conditional central limit theorem)

Suppose that assumptions hold, then for all $x \in \mathbb{R}$,

$$\mathcal{L}\left(Z_n((-\infty, \sqrt{n}x]) \mid N_n > 0\right) \xrightarrow{w} \mathcal{L}\left(\xi \mathbf{1}_{\{\mathcal{N} \leq x\}}\right), \quad n \to \infty,$$

where $\mathcal{N}$ is a standard normal random variable and independent of ξ, which is a Yaglom limit of the subcritical GW process $\{N_n; n \geq 0\}$.

Remarks

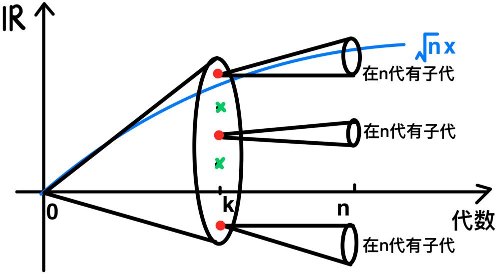
- The limit variable in Theorem 1 reflects two parts of the randomness: the Yaglom limit variable comes from the subcritical branching, and the normal variable comes from the space displacement.
- In supercritical BRW, Kaplan and Asmussen (1976) decomposed $\frac{Z_n((-\infty,\sqrt{n}x])}{m^n}$ at k_n, where $k_n = n^\beta,\ 0 < \beta < 1$. Since $N_{k_n} \to \infty$ as $n \to \infty$, they use "the law of large numbers" (see Kaplan and Asmussen (1976), Lemma 1) for the first k_n generation of BRW to get W. Since $n - k_n \to \infty$ as $n \to \infty$, they then use CLT for the last $n - k_n$ generation of BRW to get $\Phi(x)$.
- In subcritical BRW, conditioned on $\{N_n > 0\}$, N_n doesn't tend to infinity as $n \to \infty$. So we need a new decomposition.

Sketch of the proof

- **Our goal:**

$$\mathcal{L}\left(Z_n((-\infty,\sqrt{n}x]) \mid N_n > 0\right) \xrightarrow{w} \mathcal{L}\left(\xi \mathbf{1}_{\{\mathcal{N}\le x\}}\right), \quad n\to\infty, \tag{2}$$

- **Basic tools:** reduced tree, the many-to-few formula.



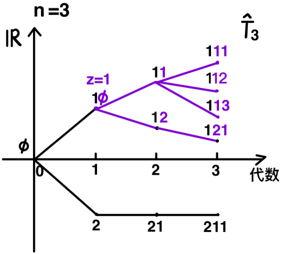
Sketch of the proof: reduced tree

- **Reduced tree**: obtained by removing all branches of the original GW tree $\mathbb{T}$, which don't extend to generation n.
- $\hat{\mathbb{T}}_n$: the reduced tree conditioned on survival up to time n, which has the root $\varnothing$.
- $\hat{\mathbb{T}}_{k,n} := \{u \in \hat{\mathbb{T}}_n : |u| = k\}$.
- $\hat{N}_{k,n} := \#\hat{\mathbb{T}}_{k,n}$, $\hat{N}_n := \hat{N}_{n,n}$.
  - $\mathcal{L}(\hat{N}_n) = \mathcal{L}(N_n \mid N_n > 0) \xrightarrow{w} \mathcal{L}(\xi)$, as $n \to \infty$.
  - $\hat{N}_{n-\tau,n}$ converges to 1 in law as n then $\tau \to \infty$.
  - Fleischmann, et al. (1977) : $\{\hat{N}_{k,n}; 0 \le k \le n\}$ is a non-homogeneous GW process, and its offspring distribution $\{p_l(\mathbf{e}_k(n))\}_{l\ge 0}$ at time k is given by

$$\sum_{l\ge 0} p_l(\mathbf{e}_k(n))s^l = \frac{f(f_{n-k-1}(0) + s\,(1 - f_{n-k-1}(0))) - f_{n-k}(0)}{1 - f_{n-k}(0)}.$$

Sketch of the proof: reduced tree

- uv: the concatenation of u and v. In particular, $\varnothing u = u\varnothing = u$.
- $z \in \hat{\mathbb{T}}_n$, the subtree $\hat{\mathbb{T}}_n$ rooted at z is defined by $\hat{\mathbb{T}}_n(z) := \{v : zv \in \hat{\mathbb{T}}_n\}$.
  - $\hat{\mathbb{T}}_{k-|z|,n}(z) := \left\{u \in \hat{\mathbb{T}}_n(z) : |zu| = k\right\}$, $|z| \le k \le n$.
  - $\hat{N}_{k-|z|,n}(z) := \#\hat{\mathbb{T}}_{k-|z|,n}$, $|z| \le k \le n$.
  - $\hat{\mathbb{T}}_n(z) := \cup_{k=|z|}^{n} \hat{\mathbb{T}}_{k-|z|,n}$
- Note $\hat{\mathbb{T}}_n(\varnothing) = \hat{\mathbb{T}}_n,\ \hat{\mathbb{T}}_{k,n}(\varnothing) = \hat{\mathbb{T}}_{k,n},\ \hat{N}_{k,n}(\varnothing) = \hat{N}_{k,n},\ \hat{N}_n(\varnothing) = \hat{N}_n$.



- **Subtree**

  $\hat{T}_3(1) = \{\phi, 1, 2, 11, 12, 13, 21\}$

- $\hat{T}_{0,3}(1) = \hat{T}_{1-1,3}(1) = \{\phi\}$ $\quad \hat{N}_{0,3}(1)=1$

  $\hat{T}_{1,3}(1) = \hat{T}_{2-1,3}(1) = \{1,2\}$ $\quad \hat{N}_{1,3}(1)=2$

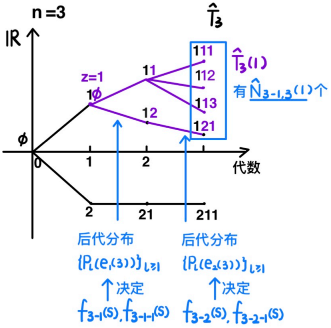
  $\hat{T}_{2,3}(1) = \hat{T}_{3-1,3}(1) = \{11,12,13,21\}$ $\quad \hat{N}_{2,3}(1)=4$

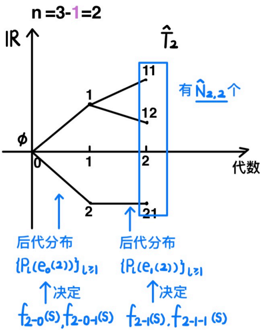
Lemma 1

For each $z \in \hat{\mathbb{T}}_n$, we have

$$\hat{N}_{n-|z|,n}(z) \overset{d}{=} \hat{N}_{n-|z|,n-|z|}(\varnothing) = \hat{N}_{n-|z|}.$$

Intuitively,





Sketch of the proof

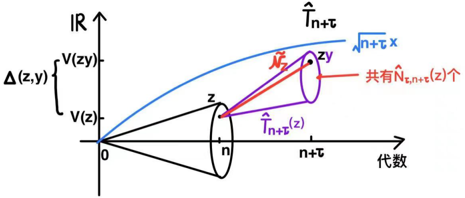
Strategy of the proof of Theorem 1 Let $\hat{Z}_n$ is a point process with law $\mathcal{L}(Z_n \mid N_n > 0)$. We decompose $\hat{Z}_{n+\tau}$ at generation n by $\hat{\mathbb{T}}_{n+\tau}$ as

$$\hat{Z}_{n+\tau}((-\infty, \sqrt{n+\tau}x]) \overset{d}{=} \sum_{z \in \hat{\mathbb{T}}_{n,n+\tau}} \left[\sum_{y \in \hat{\mathbb{T}}_{\tau,n+\tau}(z)} \mathbf{1}_{\{V(zy) \le \sqrt{n+\tau}x\}} \right]$$
$$:= A_{n+\tau} + B_{n+\tau},$$

where

$$A_{n+\tau} := \sum_{z \in \hat{\mathbb{T}}_{n,n+\tau}} \left[\left(\sum_{y \in \hat{\mathbb{T}}_{\tau,n+\tau}(z)} \mathbf{1}_{\{V(zy) \le \sqrt{n+\tau}x\}} \right) - \hat{\mathbf{N}}_{\tau,n+\tau}(z) \mathbf{1}_{\left\{ \tilde{\mathcal{N}}_z \le \sqrt{n+\tau}x - V(z) \right\}} \right],$$
$$B_{n+\tau} := \sum_{z \in \hat{\mathbb{T}}_{n,n+\tau}} \hat{N}_{\tau,n+\tau}(z) \mathbf{1}_{\left\{ \tilde{\mathcal{N}}_z \le \sqrt{n+\tau}x - V(z) \right\}},$$

and under $\mathbf{P}$, $\tilde{\mathcal{N}}_z, z \in \hat{\mathbb{T}}_{n,n+\tau}$ are mutually independent, for fixed $z, \tilde{\mathcal{N}}_z$ is distributed as $G^{\tau*}$, and independent of $\left(V(u) : u \in \hat{\mathbb{T}}_{k,n+\tau}, 0 \le k \le n \right)$ and $\hat{N}_{\tau,n+\tau}(z)$.



- $\frac{\Delta(z,y)}{\sqrt{n+\tau}} \xrightarrow[n\to\infty]{} 0$.
- each $\Delta(z,y)$ behaves as the same variable $\tilde{\mathcal{N}}_z$.

- as n first then τ tends to infinity, $\boldsymbol{A_{n+\tau} \to 0}$, thus

$$\hat{Z}_{n+\tau}((-\infty, \sqrt{n+\tau}x]) \overset{d}{\sim} B_{n+\tau} := \sum_{z\in\hat{\mathbb{T}}_{n,n+\tau}} \hat{N}_{\tau,n+\tau}(z)\mathbf{1}_{\{\tilde{\mathcal{N}}_z \le \sqrt{n+\tau}x - V(z)\}}.$$

- By Lemma 1, we have $\hat{N}_{\tau,n+\tau}(z) \overset{d}{=} \hat{N}_{\tau,\tau}(\varnothing) = \hat{N}_\tau$, and recall $\hat{N}_{n,n+\tau} \overset{d}{\sim} 1$, as n first then τ tends to infinity,

$$\boldsymbol{B_{n+\tau}} \overset{d}{\sim} \sum_{z\in\hat{\mathbb{T}}_{n,n+\tau}} \hat{N}_\tau \mathbf{1}_{\{\tilde{\mathcal{N}}_z \le \sqrt{n+\tau}x - V(z)\}} \overset{d}{\sim} \hat{N}_\tau \mathbf{1}_{\left\{\frac{V(z)}{\sqrt{n+\tau}} \le x - \frac{\tilde{\mathcal{N}}_z}{\sqrt{n+\tau}}\right\}} \overset{d}{\sim} \boldsymbol{\xi \mathbf{1}_{\{\mathcal{N}\le x\}}}. \qquad \square$$

Sketch of the proof: the asymptotic behavior of $A_{n+\tau}$

Using Lemma 1, we get

$$A_{n+\tau} \overset{d}{=} \sum_{z\in\hat{\mathbb{T}}_{n,n+\tau}} \left[\left(\sum_{y\in\hat{\mathbb{T}}_{\tau,\tau}} \mathbf{1}_{\{V(y)\le\sqrt{n+\tau}x-V(z)\}} \right) - \hat{N}_\tau \mathbf{1}_{\{\tilde{\mathcal{N}}_z\le\sqrt{n+\tau}x-V(z)\}} \right],$$

where $(V(u) : u \in \hat{\mathbb{T}}_{k,n+\tau}, 0 \le k \le n)$ is independent of $(V(u) : u \in \hat{\mathbb{T}}_\tau)$, then we have

$$A^2_{n+\tau} \overset{d}{=} \sum_{z\in\hat{\mathbb{T}}_{n,n+\tau}} \left(\sum_{y\in\hat{\mathbb{T}}_{\tau,\tau}} \mathbf{1}_{\{V(y)\le\sqrt{n+\tau}x-V(z)\}} - \hat{N}_\tau \mathbf{1}_{\{\tilde{\mathcal{N}}_z\le\sqrt{n+\tau}x-V(z)\}} \right)^2$$

$$+ \sum_{\substack{z_1,z_2\in\hat{\mathbb{T}}_{n,n+\tau}\\ z_1\neq z_2}} \prod_{i=1,2} \left(\sum_{y\in\hat{\mathbb{T}}_{\tau,\tau}} \mathbf{1}_{\{V(y)\le\sqrt{n+\tau}x-V(z_i)\}} - \hat{N}_\tau \mathbf{1}_{\{\tilde{\mathcal{N}}_{z_i}\le\sqrt{n+\tau}x-V(z_i)\}} \right).$$

(i) $\sum_{\substack{z_1,z_2\in\hat{\mathbb{T}}_{n,n+\tau}\\ z_1\neq z_2}} 1 \to 0$, as n then $\tau\to\infty$.

(ii) $\lim_{\tau\to\infty} \mathbf{E}[\hat{N}_\tau] = \frac{1}{\varphi(0)} < \infty$.

Sketch of the proof: the asymptotic behavior of $A_{n+\tau}$

$$\mathbf{E}[A_{n+\tau}^2] \overset{d}{\sim} \mathbf{E}\left[\sum_{z\in\hat{\mathbb{T}}_{n,n+\tau}}\left(\sum_{y\in\hat{\mathbb{T}}_{\tau,\tau}}\mathbf{1}_{\{V(y)\le\sqrt{n+\tau}x-V(z)\}}-\hat{N}_\tau\mathbf{1}_{\{\tilde{\mathcal{N}}_z\le\sqrt{n+\tau}x-V(z)\}}\right)^2\right]$$

$$=\mathbf{E}\left[\sum_{z\in\hat{\mathbb{T}}_{n,n+\tau}}F(n,n+\tau,V(z))\right], \quad \text{as } n \text{ then } \tau\to\infty,$$

where for $a\in\mathbb{R}$, under $\mathbf{P}$, $\tilde{\mathcal{N}}$ has the same distribution as $\tilde{\mathcal{N}}_z$ and is independent of $\mathfrak{X}_n$ and $\hat{N}_\tau$,

$$F(n,n+\tau,a)=\mathbf{E}\left[\left(\sum_{y\in\hat{\mathbb{T}}_{\tau,\tau}}\mathbf{1}_{\{V(y)\le\sqrt{n+\tau}x-a\}}-\hat{N}_\tau\mathbf{1}_{\{\tilde{\mathcal{N}}\le\sqrt{n+\tau}x-a\}}\right)^2\right]$$

$$=\mathbf{E}\left[\hat{Z}_\tau((-\infty,\sqrt{n+\tau}x-a])^2\right]-2\mathbf{E}\left[\hat{N}_\tau\sum_{y\in\hat{\mathbb{T}}_{\tau,\tau}}\mathbf{1}_{\{V(y)\le\sqrt{n+\tau}x-a\}}\mathbf{1}_{\{\tilde{\mathcal{N}}\le\sqrt{n+\tau}x-a\}}\right]$$

$$+\mathbf{E}\left[\left(\hat{N}_\tau\mathbf{1}_{\{\tilde{\mathcal{N}}\le\sqrt{n+\tau}x-a\}}\right)^2\right]$$

$$:=F_1(n,n+\tau,a)-2F_2(n,n+\tau,a)+F_3(n,n+\tau,a).$$

Lemma 2 (Many-to-one formula)

For every $n, \tau \in \{1, 2, \cdots\}$, for any $0 \le k \le n + \tau$, $x_0 \in \mathbb{R}$ and Borel measurable function g on $\mathbb{R}$, we have

$$\mathbf{E}_{\delta_{x_0}}\left[\sum_{u \in \hat{\mathbb{T}}_{k,n+\tau}} g(V(u))\right] = m^k \frac{1 - f_{n+\tau-k}(0)}{1 - f_{n+\tau}(0)} \mathbf{E}_{x_0}\left[g\left(\mathfrak{X}_k\right)\right],$$

where $\mathfrak{X} = (\mathfrak{X}_n : n \ge 0)$ is the spine random walk with the jump law G under $\mathbf{P}$.

Using Lemma 2, we obtain

$$\begin{aligned}\mathbf{E}[A_{n+\tau}^2] &\overset{d}{\sim} m^n \frac{1 - f_\tau(0)}{1 - f_{n+\tau}(0)} \cdot \mathbf{E}\left[F\left(n, n+\tau, \mathfrak{X}_n\right)\right] \\ &= \mathbf{E}\left[F_1\left(n, n+\tau, \mathfrak{X}_n\right)\right] - 2\mathbf{E}\left[F_2\left(n, n+\tau, \mathfrak{X}_n\right)\right] + \mathbf{E}\left[F_3\left(n, n+\tau, \mathfrak{X}_n\right)\right].\end{aligned}$$

Sketch of the proof: the asymptotic behavior of $A_{n+\tau}$

Lemma 3 (Second-order moment)

For every $n, \tau \in \{1, 2, \cdots\}$, for any $a \in \mathbb{R}$,

$$\begin{aligned}
&\mathbf{E}\left[\hat{Z}_\tau((-\infty, \sqrt{n+\tau}x - a])^2\right] \\
=&\frac{m^\tau}{1-f_\tau(0)}\mathbf{P}\left(\mathfrak{X}'_\tau \le \sqrt{n+\tau}x - a\right) \\
&+ \frac{m^{2\tau-1}}{1-f_\tau(0)}f''(1)\sum_{r=1}^{\tau} m^{-r}\mathbf{P}\left(\mathfrak{X}^{(1)}_{r-1,\tau} \le \sqrt{n+\tau}x - a, \mathfrak{X}^{(2)}_{r-1,\tau} \le \sqrt{n+\tau}x - a\right),
\end{aligned}$$

where under $\mathbf{P}$, for each j, $(\mathfrak{X}'_i : i \ge 0), (\mathfrak{X}^{(1)}_{j,i} : i \ge 0), (\mathfrak{X}^{(2)}_{j,i} : i \ge 0)$ are spine random walk whose jump distribution is G, and satisfying

(1) for all $i \le j, \mathfrak{X}^{(1)}_{j,i} = \mathfrak{X}^{(2)}_{j,i}$.

(2) $(\mathfrak{X}^{(1)}_{j,j+i} - \mathfrak{X}^{(1)}_{j,j} : i \ge 0)$ and $(\mathfrak{X}^{(2)}_{j,j+i} - \mathfrak{X}^{(2)}_{j,j} : i \ge 0)$ are independent.

Sketch of the proof: the asymptotic behavior of $A_{n+\tau}$

By Lemma 3 and some calculations, we can get

Proposition 1

Suppose that assumptions hold, we have

$$\limsup_{n\to\infty} \mathbf{E} A_{n+\tau}^2 \le h_1(\tau),$$

where $h_1(\tau) \to 0$ *as* $\tau \to \infty$.

Sketch of the proof: the asymptotic behavior of $B_{n+\tau}$

By some calculations, we can obtain,

$$\mathbf{E}\left[s^{B_{n+\tau}}\right]=\mathbf{E}\left[s^{\sum_{z\in\hat{\mathbb{T}}_{n,n+\tau}}\hat{N}_\tau(z)\mathbf{1}_{\{\widetilde{\mathcal{N}}_z\le\sqrt{n+\tau}x-V(z)\}}}\right]=\mathbf{E}\left[\prod_{z\in\hat{\mathbb{T}}_{n,n+\tau}}R(n,n+\tau,V(z))\right],$$

where for $a\in\mathbb{R}$, $s\in[0,1]$,

$$R(n,n+\tau,a)=\mathbf{P}\left(\tilde{\mathcal{N}}_z\le\sqrt{n+\tau}x-a\right)\mathbf{E}\left[s^{\hat{N}_\tau}\right]+1-\mathbf{P}\left(\tilde{\mathcal{N}}_z\le\sqrt{n+\tau}x-a\right).$$

Then, for fixed $z_0\in\hat{\mathbb{T}}_{n,n+\tau}$,

(i) Using the fact that $\left|\prod_{m=1}^n z_m-\prod_{m=1}^n w_m\right|\le\theta^{n-1}\sum_{m=1}^n|z_m-w_m|$, where $|z_i|\le 1,|w_i|\le 1$, then

$$\left|\mathbf{E}\left[s^{B_{n+\tau}}\right]-\mathbf{E}[R\left(n,n+\tau,V\left(z_0\right)\right)\cdot\prod_{\substack{z\in\hat{\mathbb{T}}_{n,n+\tau}\\ z\neq z_0}}1]\right|\le\mathbf{E}[\sum_{\substack{z\in\hat{\mathbb{T}}_{n,n+\tau}\\ z\neq z_0}}(1-R(n,n+\tau,V(z_0)))]$$

$$\le\mathbf{E}\left[\hat{N}_{n,n+\tau}-1\right]\to\frac{1-f_\tau(0)}{\varphi(0)m^\tau}-1,\ \ n\to\infty.$$

Sketch of the proof: the asymptotic behavior of $B_{n+\tau}$

(ii) Since the law of $V(z_0)$ is G^{n*}, and $V(z_0)$ is indepnedent of $\widetilde{\mathcal{N}}_{z_0}$, then

$$\mathbf{E}\left[R\left(n, n+\tau, V\left(z_0\right)\right)\right] \rightarrow \Phi(x)\mathbf{E}\left[s^{\hat{N}_\tau}\right]+1-\Phi(x),$$

whose limit is equal to $\mathbf{E}\left[s^{\xi \mathbf{1}_{\{\mathcal{N} \leq x\}}}\right]$, as $\tau \rightarrow \infty$.

Proposition 2

Suppose that assumptions hold, then for all $s \in [0,1]$, we have

$$\limsup_{n \rightarrow \infty}\left|\mathbf{E}\left[\mathrm{s}^{B_{n+\tau}}\right]-\mathbf{E}\left[s^{\xi \mathbf{1}_{\{\mathcal{N} \leq x\}}}\right]\right| \leq h_2(\tau, s),$$

where $h_2(\tau, s) \rightarrow 0$ as $\tau \rightarrow \infty$.

Applications: Conditional limit theorems

Corollary

Suppose that assumptions hold, we have, for fixed $l \geq 1$, for all $x \in \mathbb{R}$,
(i) For $j \in \mathbb{N}$,

$$\lim_{n\to\infty} \mathbf{P}\left(Z_n((-\infty, \sqrt{n}x]) = j \mid N_{n+l} > 0\right) = b_j(l; x) \geq 0$$

where $\sum_{j\geq 0} b_j(l; x) = 1$. Recall $\mathscr{B}(s) = \mathbf{E}\left[s^{\xi}\right]$, then the generating function of $\{b_j(l; x)\}_{j\geq 0}$ is defined by

$$\frac{1}{m^l}\left\{\Phi(x)\left[\mathscr{B}(s) - \mathscr{B}\left(sf_l(0)\right)\right] + (1 - \Phi(x))\left[1 - \mathscr{B}\left(f_l(0)\right)\right]\right\}.$$

(ii) As $l \to \infty$ and then $n \to \infty$,

$$\mathcal{L}\left(Z_n((-\infty, \sqrt{n}x]) \mid N_{n+l} > 0\right) \xrightarrow{w} \mathcal{L}\left(\zeta \mathbf{1}_{\{\mathcal{N} \leq x\}}\right),$$

where the standard normal random variable $\mathcal{N}$ is independent of random variable ζ with distribution $\{\pi_j\}_{j\geq 1}$.

Applications: Conditional limit theorems

Remark

The assumption of G can be generalized to the case that G is in the domain of attraction of a stable law, that is, there exists a non-degenerate random variable η, $\{a_n\} \subset \mathbb{R}, \{b_n\} \subset \mathbb{R}^+$ such that, as $n \to \infty$, $\boldsymbol{(S_n - a_n)/b_n \xrightarrow{d} \eta}$, where S_n is the sum of n independent identically distributed variables with law G. Under this assumption, we only need to replace $(-\infty, \sqrt{n}x]$, $\mathcal{N}$ and $\Phi(x)$ with $(-\infty, b_n x + a_n]$, η and $\mathbf{P}(\eta \leq x)$ respectively in the above results.

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Thank you!